INTERNATIONAL BACCALAUREATE

Mathematics: applications and interpretation

MAI

EXERCISES [MAI 1.9-1.11]

COMPLEX NUMBERS (POLAR FORM)

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A. Paper 1 questions (SHORT)

1. [Maximum mark: 8] *[try to answer without GDC]*

Write down the polar form of the following complex numbers

Cartesian form	**Polar form**		
$x + y\text{i}$	$r\text{cis}\,\theta$	$r\left(\cos\theta + \text{i}\sin\theta\right)$	$r\text{e}^{\text{i}\theta}$
1			
-1			
i			
$-\text{i}$			
8			
-8			
8i			
-8i			

2. [Maximum mark: 4] *[try to answer without GDC]*

Write down the Cartesian form of the following complex numbers

Polar form	**Cartesian form**
$4\text{cis}0$	
$5\text{cis}\,\pi$	

Polar form	**Cartesian form**
$6\text{cis}\,\frac{\pi}{2}$	
$7\text{cis}\left(-\frac{\pi}{2}\right)$	

3. [Maximum mark: 6]

Find the polar form $r\text{cis}\theta$ of the complex numbers

$$z_1 = 1+\text{i}, \qquad z_2 = -1+\text{i}, \qquad z_3 = -1-\text{i}, \qquad z_4 = 1-\text{i}$$

and represent them on the complex plane.

4. [Maximum mark: 6]

Let $z = \sqrt{3} + \mathrm{i}$

(a) Find the polar form $r\mathrm{cis}\theta$ of the complex numbers z and $-z$. [2]

(b) Find the polar form $r\mathrm{cis}\theta$ of the complex numbers $\overline{z}$ and $-\overline{z}$. [2]

(c) Represent the four complex numbers above on the Complex plane. [2]

5. [Maximum mark: 8] *[try to answer without GDC]*

Let $z = 2\text{cis}20°$ and $w = 6\text{cis}40°$

(a) Express the following results in the polar form $r\text{cis}\theta$

zw	
$\frac{w}{z}$	
$\frac{z}{w}$	
z^2	
z^3	

[5]

(b) Express $zw + z^3$ in polar and **hence** in Cartesian form $x + y\text{i}$. [3]

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6. [Maximum mark: 4]

The complex number z is defined by

$$z = 4\left(\cos\frac{2\pi}{3} + \text{i}\sin\frac{2\pi}{3}\right) + 4\sqrt{3}\left(\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6}\right)$$

Express z in the form $re^{i\theta}$.

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7. [Maximum mark: 7]

(a) Find the polar form for each of the complex numbers

$z = 1 + i\sqrt{3}$ $\quad w = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}$ $\quad u = -2\sqrt{3} + 2\mathrm{i}$ [3]

(b) Then find the polar form of

(i) zw (ii) $\frac{z}{w}$ (iii) u^2 (iv) zwu [4]

[use degrees for the argument]

8. [Maximum mark: 8]

Let $z_1 = \frac{\sqrt{6} - i\sqrt{2}}{2}$, and $z_2 = 1 - i$.

(a) Find the value of $\frac{z_1}{z_2}$ in the form $a + bi$, [2]

(a) Write z_1 and z_2 in the form $r\text{cis}\theta$, where $r > 0$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. [2]

(b) Show that $\frac{z_1}{z_2} = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$. [2]

(c) **Hence** find the exact values of cos $\frac{\pi}{12}$ and sin $\frac{\pi}{12}$. Justify your answer. [2]

9. [Maximum mark: 4]

Let $z_1 = a\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$ and $z_2 = b\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$

Express $\left(\frac{z_2}{z_1}\right)^2$ in Cartesian form.

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10. [Maximum mark: 9]

Let $z = r\mathrm{cis}\theta$ and. $w = 3\mathrm{cis}\frac{\pi}{5}$

(a) Express in terms of r and θ the following

(i) zw (ii) $\frac{z}{w}$ (ii) zw^2 [6]

(b) Describe geometrically the effect of w on z, for the operations in question (a). [3]

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11. [Maximum mark: 6]

Given that $z = (b + \mathrm{i})^2$, where b is real and positive, find the **exact** value of b when $\arg z = 60°$.

METHOD A: Expand $(b + \mathrm{i})^2$ and then consider the argument of z.

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METHOD B: Think of the argument of $b + \mathrm{i}$ first.

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12. [Maximum mark: 4]

Let

$$z = 2\text{cis}(x+1) = 2e^{(x+1)i} = 2[\cos(x+1) + i\sin(x+1)]$$

$$w = 3\text{cis}(x+2) = 3e^{(x+2)i} = 3[\cos(x+2) + i\sin(x+2)]$$

Write down the three corresponding expressions of the polar form for the product zw.

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13. [Maximum mark: 7]

Let

$$z = 2\text{cis}(0.2x) = 2e^{0.2xi} = 2[\cos(0.2x) + i\sin(0.2x)]$$

$$w = 8\text{cis}(0.8x) = 8e^{0.8xi} = 8[\cos(0.8x) + i\sin(0.8x)]$$

Write down the three corresponding expressions of the polar form

(a) for the product zw. [4]

(b) For the quotient $\frac{w}{z}$ [3]

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14. [Maximum mark: 7]

Let $z = 2e^{(2x+1)i}$ and $w = 3e^{(2x+3)i}$

(a) Find $z + w$ in the form $Ae^{(2x+B)i}$ [5]

(b) Hence express

(i) $2\cos(2x+1) + 3\cos(2x+3)$ in the form $A\cos(2x+B)$

(ii) $2\sin(2x+1) + 3\sin(2x+3)$ in the form $A\sin(2x+B)$ [2]

15. [Maximum mark: 7]

Let $z = 3e^{(0.2x+1)i}$ and $w = 5e^{0.2xi}$

(a) Find $z + w$ in the form $Ae^{(0.2x+B)i}$ [5]

(b) Hence express

(i) $3\cos(0.2x+1) + 5\cos(0.2x)$ in the form $A\cos(0.2x+B)$

(ii) $3\sin(0.2x+1) + 5\sin(0.2x)$ in the form $A\sin(0.2x+B)$ [2]

16. [Maximum mark: 8]

Let $A(t) = 10\cos(3t + 4)$ and $B(t) = 20\cos(3t + 5)$

$A(t)$ and $B(t)$ are the real parts of the complex numbers z_A and z_B respectively.

(a) Express z_A and z_B in the form $r\text{cis}\,\theta$ and $re^{i\theta}$, where θ is in terms of t. [2]

(b) Find $z_A + z_B$ in the form $re^{i\theta}$ where θ is in terms of t. [5]

(c) Hence, express $A(t) + B(t)$ in the form $a\cos(3t + b)$ [1]

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17. [Maximum mark: 10]

Let $A(t) = 7\sin(0.7t)$ and $B(t) = \sin(0.7t + 5)$

Find $A(t) + B(t)$ in the form $a\sin(0.7t + b)$ by using two different methods

(a) Using complex numbers. [6]

(b) Using trigonometry, by observing the graph of the function $A(t) + B(t)$. [4]

18. [Maximum mark: 10]

Let $A(t) = 5.1\sin(\omega t)$ and $B(t) = 3.2\sin(\omega t + 5)$

Find $A(t) + B(t)$ in the form $a\sin(\omega t + b)$ by using two different methods

(a) Using complex numbers. [6]

(b) Using trigonometry, by observing the graph of $y = 5.1\sin x + 3.2\sin(x + 5)$. [4]

B. Paper 2 questions (LONG)

19. [Maximum mark: 12]

(a) Find the polar form of the complex numbers $z = 1 + i$ and $w = \sqrt{3} - i$ [2]

(b) Find zw by using the Cartesian forms. [2]

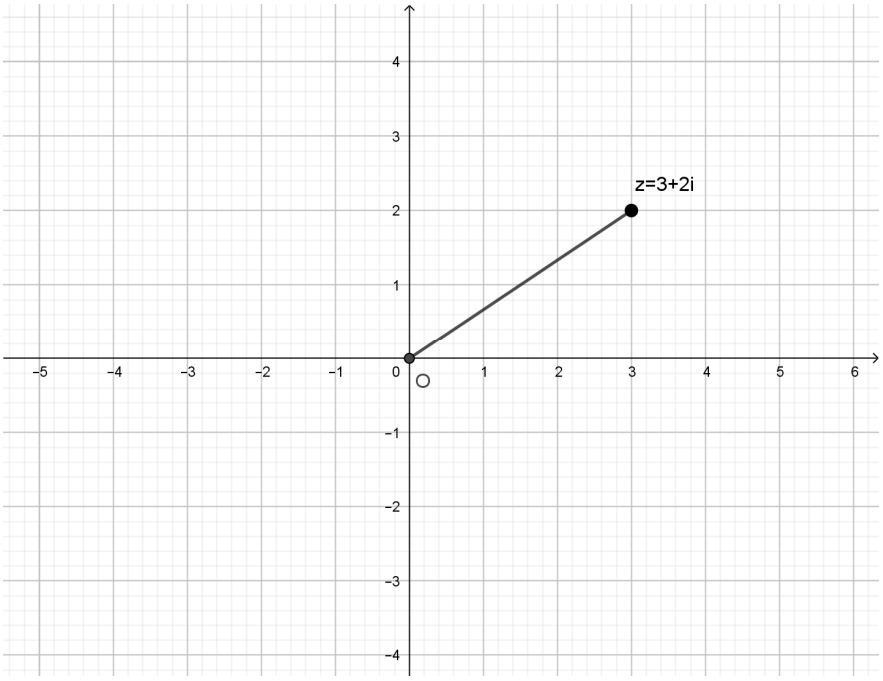
(c) Find zw by using the polar forms. [2]

(d) Deduce the exact values of $\cos 15^\circ$, $\sin 15^\circ$ and $\tan 15^\circ$ [6]

[use degrees for the arguments]

20. [Maximum mark: 11]

Let $z = 2 + 3\mathrm{i}$



(a) Find $z\mathrm{i}$, $z\mathrm{i}^2$ and $z\mathrm{i}^3$ and represent them on the diagram above. [4]

The Euler polar form of z is $r\mathrm{e}^{\mathrm{i}\theta}$ where $\tan\theta = \frac{2}{3}$, $0 < \theta < \frac{\pi}{2}$.

(b) Find r [2]

(c) Express $z\mathrm{i}$, $z\mathrm{i}^2$ and $z\mathrm{i}^3$ in Euler form in terms of θ and describe geometrically the result. [5]

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21. [Maximum mark: 10]

A complex number $z = x + y\mathrm{i}$ is such that $|z| = |z - 3\mathrm{i}|$.

(a) Show that the imaginary part of z is $\frac{3}{2}$. [3]

(b) Let z_1 and z_2 be the two possible values of z, such that $|z| = 3$.

(i) Sketch a diagram to show the points which represent z_1 and z_2 in the complex plane, where z_1 is in the first quadrant.

(ii) Show that $\arg z_1 = \frac{\pi}{6}$.

(iii) Find $\arg z_2$.

(iv) Write down the values of z_1 and z_2 in polar form. [7]

22. [Maximum mark: 15]

Let $y = \cos\theta + \mathrm{i}\sin\theta$

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}\theta} = \mathrm{i}y$ *[i may be treated in the same way as a real constant]* [3]

(b) Find the value of y and the value of $\frac{\mathrm{d}y}{\mathrm{d}\theta}$, when $\theta = 0$ [2]

(c) Solve the differential equation in (a) to show that $y = \mathrm{e}^{\mathrm{i}\theta}$. [5]

(d) Use this result to deduce that

(i) $(\cos\theta + \mathrm{i}\sin\theta)(\cos\phi + \mathrm{i}\sin\phi) = \cos(\theta + \phi) + \mathrm{i}\sin(\theta + \phi)$

(ii) $\frac{\cos\theta + \mathrm{i}\sin\theta}{\cos\phi + \mathrm{i}\sin\phi} = \cos(\theta - \phi) + \mathrm{i}\sin(\theta - \phi)$

(iii) $(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta$ [5]

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